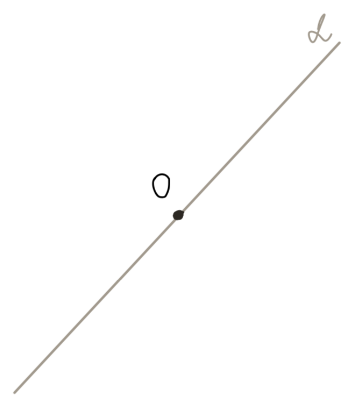
Linear Transformation

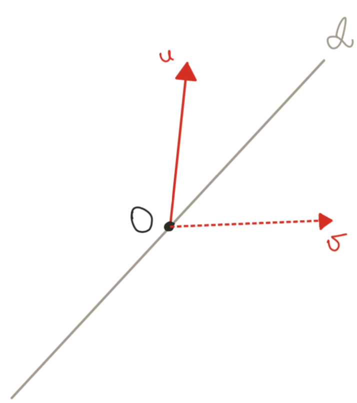
Before delving into the rigorous definition of linear transformation I would like to consider the following example.

Example:

I want you to imagine an origin *O* and an arbitrary line α passing through it:



We already know that in the context of this tutorial all vectors must start at the origin so I propose you to imagine a random vector *u* that starts at the origin. Next, I would like you to reflect *u* over the line α and call the resulting vector *v*. Here is how I did it:

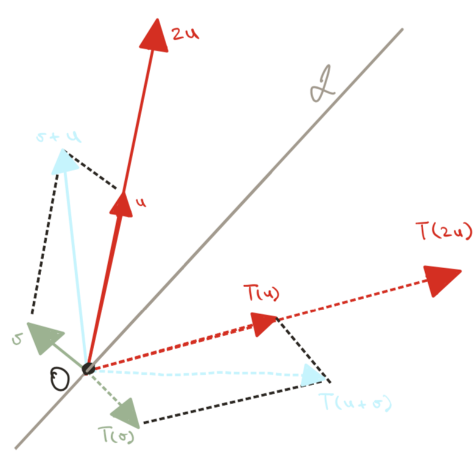


Now I would like you to think of the reflection procedure above as a function *T*. So that T(u) = v. Let's play with T for a little while. I propose to make vector u twice longer (in fancy math terms: scale u by the factor of 2) and calculate its reflection. Not surprisingly, the resulting vector is twice longer than v. So we can represent this observation mathematically in the following way: T(2**u**) = 2**v.** In other words, it does not matter whether we scale u and then reflect it or reflect it and then scale the result. See the figure below.

I think that it is easy to see that if 2 is replaced with some random real number c then the equation will still hold. More generally: T(c**u**) = c**v**.

Note: The real number c can also be called a *scalar.*

Now a more difficult scenario. Imagine 2 vectors u and v. Add them together to get vector w and then reflect vector w over line α. See the figure below. Now, instead of adding u and v and then reflecting the result I want you to reflect u, then reflect v and then sum the reflected versions of u and v. I did it the following way:



Did you notice that we get the same result? I claim that this is not at all a coincidence and this holds for all possible vectors (you can try more if you don't believe me). This observation can be stated mathematically in the following way:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

In other words, it does not matter whether we add **u** and **v** and then reflect the result or reflect **u** and **v** separately and then add the reflected versions together.

> Definition:
>
> Any function T that satisfies both observations mentioned above:
>
> $$T(c\mathbf{u}) = c\mathbf{v}$$
>
> $$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
>
> is called a *linear transformation*.

So the reflection that was discussed is a linear transformation. You may wonder: How many different linear transformations except reflection are there? I will answer this question in a section devoted to matrix representation of linear transformations.